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Note

Chromatic difference sequences and homomorphisms

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Abstract

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This note characterizes graphs with the second term of their chromatic difference sequences equal to 1, and gives a class of graphs, called W_n, that are determined by their chromatic difference sequences. It also gives a large class of n-chromatic graphs for which W_n is a homomorphic image. It is proved that the normalized chromatic difference sequence of the categorical product $G \times H$ dominates the normalized chromatic difference sequence of G and of H.

1. Introduction

We only consider finite undirected graphs without loops and multi-edges in this paper. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. The *chromatic difference sequence* of a graph G with chromatic number n is defined by $\text{cds}(G) = (a(1), a(2), \ldots, a(n))$ where $\sum_{i=1}^{t} a(i)$ is the number of vertices in a maximum t-colorable induced subgraph of G [1, 2]. The *normalized chromatic difference sequence* of a graph G is defined by $\text{ncds}(G) = \text{cds}(G)/|V(G)|$. A *homomorphism* f from a graph G to a graph H is a mapping $f: V(G) \rightarrow V(H)$ such that $f(x)f(y) \in E(H)$ wherever $xy \in E(G)$. If in addition f is onto $V(H)$ and each edge in H is the image of some edge in G, then f is said to be *onto* and H is called a *homomorphic image* of G. A sequence $(x_1, \ldots, x_n)$ is said to *dominate* the sequence $(y_1, \ldots, y_n)$, denoted by

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$(x_1, \ldots, x_n) \geqslant^* (y_1, \ldots, y_n)$, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, and $\sum_{i=1}^p x_i \geqslant \sum_{i=1}^p y_i$ for $p = 1, 2, \ldots, n-1$. Although domination is usually defined for sequences with the same number of terms. It is natural to extend the definition of domination to two sequences with a different number of nonzero terms. A sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$ is said to dominate the sequence $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$ if $n \leqslant m$, and $\boldsymbol{x}' = (x_1, \ldots, x_n, 0, \ldots, 0)$, where there are $m-n$ zeros, dominates $\boldsymbol{y}$. It is also easy to see that when $x_n \neq 0$, $n > m$ then $\boldsymbol{x}$ can not dominate $\boldsymbol{y}$.

We already know that any sequence with n nonincreasing terms is the cds of a complete n-partite graph [1]. We now consider those sequences with one term equal to 1, and determine which graphs have these sequences as their cds. Theorem 6 gives us an interesting example: the chromatic difference sequence uniquely determines a graph. Finally Theorem 9 gives us a typical example of characterizing the homomorphic image of G from the properties of cds of G. In general a homomorphism from a graph G to (or onto) another graph W need not induce a domination between the ncds of G and W. However under certain conditions a homomorphism from a graph G to another graph W will imply that ncds(G) dominates ncds(H). For example, if there exists a homomorphism from G to W and W is vertex-transitive, then ncds(G) dominates ncds(W) [2]. Another example involves the Categorical product of two graphs which is defined as follows. The Categorical product $G \times W$ [4] has as vertex set $V(G) \times V(H)$ and has as edge set the edges $(g, w)(g', w')$ where $gg' \in E(G)$ and $ww' \in E(W)$. The projection mapping from $G \times W$ to G is a homomorphism and ncds($G \times W$) dominates ncds(G) (Theorem 10), from which it is easy to show that $\lim_{k \to \infty}$ ncds(G^k) exists, where G^k is recursively defined by $G^1 = G$, $G^k = G^{k-1} \times G$ (Corollary 11). For the Cartesian powers of a graph, we have similar results [5]. The limit behavior of some other graph theoretical parameters of three different graph products were considered in [3].

2. Main results

Lemma 1. *Assume that a graph G of chromatic number n has* cds(G) $= (a(1), \ldots, a(n))$ *with $a(t) = 1$ where $t \leqslant n$. Then:*

(1) *$V(G) = A \cup B$, where A induces a maximum $(t-1)$-colorable subgraph with $a(1) + \cdots + a(t-1)$ vertices and B induces a complete subgraph with $a(t) + \cdots + a(n) \leqslant n$ vertices.*

(2) *If $A_1, A_2, \ldots, A_{t-1}$ is a $t-1$ coloring of A, then for any $y \in B$, there exists at least one edge between y and each A_i $(i = 1, 2, \ldots, t-1)$.*

(3) *$a(s) \leqslant t$ for $s = t+1, t+2, \ldots, n$.*

Remark. If $t = 1$, then $A = \emptyset$. Hence G is a complete graph with n vertices.

Proof. (1) Let A be the vertex set of a maximum $(t-1)$-coloarable subgraph. Then $|A| = a(1) + \cdots + a(t-1)$. Let $A_1, \ldots, A_{t-1}$ be a $t-1$ coloring of A and let $B = V(G) - A$. Thus B induces a complete subgraph with $a(t) + \cdots + a(n) \leq n$ vertices, for otherwise $a(t) \geq 2$.

(2) Now if for $y_0 \in B$ and some $i \in \{1, \ldots, t-1\}$, there is no edge between y_0 and A_i, then $A_1 \cup \cdots \cup A_{i-1} \cup [A_i \cup \{y_0\}] \cup \cdots \cup A_{t-1}$ induces a $(t-1)$-colorable subgraph of G with $a(1) + \cdots + a(t-1) + 1$ vertices, a contradiction.

(3) Suppose, otherwise, that $a(j) > t$ for some $j \in \{t+1, t+2, \ldots, n\}$. Since $a(i) > 0$ for any $i \in \{1, 2, \ldots, n\}$, we have $\sum_{i=t}^{n} a(i) = \sum_{i=t, i \neq j}^{n} a(i) + a(j) > n - t + t = n$, a contradiction. $\square$

Lemma 2. *Assume that a graph G of chromatic number n has* $\mathrm{cds}(G) = (a(1), \ldots, a(n))$ *with $a(2) = 1$. Then G and its* cds *is either in Case* 1 *or in Case* 2.

Case 1: (1) $a(3) = \cdots = a(n) = 1$; *and*

(2) *$V(G) = A \cup B$ where $|A| = a(1) - 1$, $|B| = n$, A is an independent set and B induces a complete subgraph. There exists a vertex x_0 in B such that there is no edge between this vertex and A but for all other vertices x of B there exists at least one edge between x and A.*

Case 2: (1) *every one of $a(3), \ldots, a(n)$ is equal to* 1, *except one, say $a(t)$ $(t \in \{3, 4, \ldots, n\})$ which is equal to* 2;

(2) *$V(G) = A \cup B$, where A is an independent set with $a(1)$ vertices, B induces a complete subgraph with n vertices, for any $y \in B$ there is at least one edge between y and A; and*

(3) *if $a(t) = 2$ $(t \in \{3, 4, \ldots, n\})$, then there exists a subset $B_1 \subseteq B$, $|B_1| = t$ such that*:

(3.1) *for any $x \in A$, there exists a $y \in B_1$ such that there is no edge between x and y; and*

(3.2) *for any $t \in B_1$, there exists an $x \in A$ such that there is no edge between x and y.*

Proof. By Lemma 1, $V(G) = A^* \cup B^*$ where A^* is a maximum independent set with $a(1)$ vertices, B^* induces a complete subgraph with $a(2) + \cdots + a(n)$ vertices, and for any $y \in B^*$ there exists at least one edge between y and A^*.

Since $a(2) + \cdots + a(n) \leq n$, and $a(i) \geq 1$, for $i = 1, \ldots, n$, then we have either $a(2) + \cdots + a(n) = n - 1$ or $a(2) + \cdots + a(n) = n$.

Case 1: (1) If $a(2) + \cdots + a(n) = n - 1$, then $a(2) = \cdots = a(n) = 1$ and $|B^*| = n - 1$.

(2) We color B^* by $n - 1$ different colors. If for any $x \in A^*$, there exists a $y \in B^*$ such that there is no edge between x and y, then we can color x by the color of y. Thus G is $(n-1)$-colorable, a contradiction. Therefore there exists an $x_0 \in A^*$ which is joined to every y in B^*. Let $A = A^* - \{x_0\}$, $B = B^* \cup \{x_0\}$.

Case 2: (1) If $a(2) + \cdots + a(n) = n$, then every one of $a(3), \ldots, a(n)$ is equal to 1, except one, say $a(t)$ $(t \in \{3, 4, \ldots, n\})$ which is equal to 2. ($a(2) = 1$ is given by the assumption.)

(2) Let $A = A^*$, $B = B^*$.

(3) Assume that $a(t)=2$ where $t\in\{3,4,\ldots,n\}$. By the definition of cds, a maximum $(t-1)$-colorable subgraph has $a(1)+t-2$ vertices. The maximum t-colorable subgraph has $a(1)+t$ vertices. Let H be the maximum t-colorable subgraph. Let $B_1=B\cap V(H)$. If $|B_1|>t$, then H cannot be t-colorable. If $|B_1|<t$, then $|V(H)|<a(1)+t$, hence H is not the maximum t-colorable subgraph. Therefore $|B_1|=t$, and $V(H)=A\cup B_1$.

(3.1) If for some $x_0\in A$ we have that $x_0y\in E(G)$ for every $y\in B_1$, then the subgraph of H induced by $\{x_0\}\cup B_1$ is a $(t+1)$-complete subgraph. This contradicts the fact that $\chi(H)=t$.

(3.2) Now suppose that $y_0\in B_1$ with $y_0x\in E(G)$ for every $x\in A$. Consider the subgraph $H-y_0$. The $t-1$ vertices of B_1-y_0 can be colored by $t-1$ different colors. For any $x\in A$, by (3.1) there exists a $y\in B_1$ such that $xy\notin E(G)$. y cannot be y_0, therefore we can color x by the color of y. Thus $H-y_0$, which has $a(1)+t-1$ vertices, is $(t-1)$-colorable. This contradicts the fact that $a(1)+\cdots+a(t-1)=a(1)+t-2$. □

Corollary 3. (1) *Any sequence $\boldsymbol{r}=(r_1,r_2,\ldots,r_n)$ with $r_2=1$, some $r_t>2$ for $t\in\{3,\ldots,n\}$ can not be the choromatic difference sequence of some graph.*

(2) *Any sequence $\boldsymbol{r}=(r_1,r_2,\ldots,r_n)$ with $r_2=1$, $r_{t_1}=r_{t_2}=2$ for some $t_1,t_2\in\{3,\ldots,n\}$ can not be the chromatic difference sequence of some graph.*

This corollary can also be derived from [1] as mentioned by the referee.

Definition 4. Let W_n be a graph with $2n$ vertices: $a_1',a_2',\ldots,a_n',a_1,a_2,\ldots,a_n$. The vertices $a_1',a_2',\ldots,a_n'$ induce a complete graph K_n and the set of vertices $\{a_1,a_2,\ldots,a_n\}$ form an independent set. There are edges between a_i' and a_j $(i=1,2,\ldots,n;j=1,2,\ldots,n;i\neq j)$, no edge between a_i and a_i' $(i=1,\ldots,n)$.

Lemma 5 is not hard to prove.

Lemma 5. $\text{cds}(W_n)=(n,1,1,\ldots,1,2)$ *with n terms.*

Theorem 6. *A graph G is isomorphic to W_n if and only if* $\text{cds}(G)=(n,1,\ldots,1,2)$ *with n terms.*

Proof. Obviously two isomorphic graphs have the same cds. Thus $\text{cds}(G)=\text{cds}(W_n)=(n,1,\ldots,1,2)$ with n terms by Lemma 5 if G is isomorphic to W_n. Conversely if $\text{cds}(G)=(n,1,\ldots,1,2)$ with n terms, then we shall prove that G is isomorphic to W_n. By Case 2 of Lemma 2, $t=n$, $V(G)=A\cup B$, where A is an independent set with n vertices, B induces a complete subgraph with n vertices. For any $x\in A$, there exists $y\in B$ such that there is no edge between x and y. For any $y\in B$, there exists $x\in A$ such that there is no edge between y and x.

Claim 1. *For any $y \in B$, there exists exactly one $x \in A$ such that there is no edge between y and x.*

Proof. Suppose, otherwise, that for $y_0 \in B$, there are two $x_1, x_2 \in A$ such that there is no edge from y_0 to x_1 and x_2. For any $x \in A - \{x_1, x_2\}$ take $\tau(x) \in B$ such that there is no edge between x and $\tau(x)$. Let $B_1 = \{\tau(x): x \in A - \{x_1, x_2\}\}$. Then $|B_1| \leqslant n-2$. Let H be the subgraph induced by $A \cup \{y_0\} \cup B_1$. If $y_0 \notin B_1$, then H can be colored by $|B_1| + 1 \leqslant n-1$ colors. But H has $n + |B| + 1$ vertices. This contradicts the fact that t-colorable $(t < n)$ subgraphs of G can have at most $n+t-1$ vertices. If $y_0 \in B_1$, then H can be colored by $|B_1| \leqslant n-2$ colors. But H has $n + |B_1|$ vertices, which is a contradiction. □

Claim 2. *For any $x \in A$, there exists exactly one $y \in B$ such that there is no edge between y and x.*

Since for any $y \in B$, there exists exactly one $x \in A$ such that there is no edge between y and x, we must have $n(n-1)$ edges between B and A. We already know that for any $x \in A$, there exists a $y \in B$ such that there is no edge between x and y, i.e., for any $x \in A$ there is at most $n-1$ edges between x and B. Now in order that there are $n(n-1)$ edges between B and A, for any $x \in A$ we must have exactly $n-1$ edges between x and B. Therefore for any $x \in A$, there is exactly one $y \in B$ such that there is no edge between x and y.

Therefore there is a one to one correspondence between the vertices of A and B. Let $A = \{a_1, a_2, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ be such that there is no edge between a_i and b_i $(i = 1, 2, \ldots, n)$, then there exists an edge between a_i and b_j for any $i = 1, 2, \ldots, n$, any $j = 1, 2, \ldots, n$, with $i \neq j$. Thus G is isomorphic to W_n. □(Theorem 6)

Corollary 7. *If* $\operatorname{cds}(G) = (t, 1, \ldots, 1, 2, 1, \ldots, 1)$ *where 2 is in the t-th position, then G contains an induced subgraph which is isomorphic to W_t.*

Lemma 8. *Let G be a graph and* $\operatorname{cds}(G) = (a(1), a(2), \ldots, a(n))$. *Then $a(1) \geqslant a(i)$ for $i = 2, 3, \ldots, n$.*

Proof. Suppose $a(1) < a(t)$ for some $t \in \{2, 3, \ldots, n\}$. By definition the maximum $(t-1)$-colorable induced subgraph of G has $a(1) + \cdots + a(t-1)$ vertices. The maximum t-colorable induced subgraph, say G_t, of G has $a(1) + a(2) + \cdots + a(t-1) + a(t)$ vertices. Let $H_1, H_2, \ldots, H_t$ be the color classes of a t-coloring of G_t. Then H_1 is an independent set of G_t, also an independent set of G, hence $|H_1| \leqslant a(1)$. Thus $|G_t - H_1| \geqslant a(2) + \cdots + a(t-1) + a(t) > a(2) + \cdots + a(t-1) + a(1)$. $G_t - H_1$ is a $(t-1)$-colorable subgraph of G, which contradicts to the fact that the maximum $(t-1)$-colorable subgraph has $a(1) + \cdots + a(t-1)$ vertices. □

Theorem 9. *Let G be a graph with* $\text{cds}(G) = (a(1), a(2), \ldots, a(n))$ *and* $a(n) > a(i)$ *for* $i = 2, 3, \ldots, n-1$. *Then W_n is a homomorphic image of G.*

Proof. Let I be a maximum independence set of G and let $H = G - I$. Let $A_1, A_2, \ldots, A_n$ be the color classes in an n-coloring of G. Let $I_i = I \cap A_i$ and let $H_i = H \cap A_i$ $(i = 1, \ldots, n)$ and mapping f by

$$f(x) = a_i \ \text{for } \forall x \in I_i; \qquad f(x) = a_i' \ \text{for } \forall x \in H_i.$$

We shall show that f is a homomorphism from G onto W_n.

Claim 1. *f is a homomorphism from H onto the n-complete subgraph induced by* $\{a_1', \ldots, a_n'\}$.

Proof. To show that H is not $(n-1)$-colorable assume the contrary. Let U_i $(i = 1, \ldots, n-1)$ be the $n-1$ color classes of a $(n-1)$-coloring of H and let $|U_i| \geqslant |U_{i+1}|$ $(i = 1, \ldots, n-2)$. Then I and $U_1, \ldots, U_{i-2}$ determine a $(n-1)$-colorable subgraph of G. Hence

$$a(1) + \cdots + a(n-1) \geqslant a(1) + |U_1| + \cdots + |U_{n-2}|,$$

i.e., $a(n) \leqslant |U_{n-1}|$ and $a(2) + \cdots + a(n-1) \geqslant |U_1| + \cdots + |U_{n-2}|$, so

$$(n-2)a(n) \leqslant (n-2)\,|U_{n-1}| \leqslant |U_1| + \cdots + |U_{n-2}| \leqslant a(2) + \cdots + a(n-1),$$

i.e. $a(n) \leqslant a(i)$ holds for at least one i $(i = 2, \ldots, n-1)$, which is a contradiction. □

Thus H is n-chromatic since H is n-colorable. Therefore each A_i $(i = 1, 2, \ldots, n)$ is non-empty. Furthermore any edge in the complete subgraph induced by $\{a_1', \ldots, a_n'\}$ must be the image of edges of H, otherwise H would be $(n-1)$-colorable.

Claim 2. *Each I_i $(i = 1, 2, \ldots, n)$ is non-empty.*

Proof. Suppose without loss of generality that $I_n = \emptyset$. We know that $a(1) = \sum_{i=1}^{n-1} |I_i|$ since I is a maximum independence set. Since $\sum_{i=1}^{n-1} a(i) \geqslant \sum_{i=1}^{n-1} |I_i| + \sum_{i=1}^{n-1} |H_i|$, we have $a(2) + \cdots + a(n-1) \geqslant |H_1| + \cdots + |H_{n-1}|$. One the other hand, $a(1) + a(2) \geqslant \sum_{i=1}^{n-1} |I_i| + |H_n|$, hence we have $a(2) \geqslant |H_n|$. Thus

$$\sum_{i=1}^{n} |H_i| = a(2) + \cdots + a(n-1) + a(n) > a(2) + \cdots + a(n-1) + a(2)$$

$$\geqslant \sum_{i=1}^{n} |H_i|,$$

a contradiction. Hence $I_n = \emptyset$. Similarly $I_i \neq \emptyset$ for $i = 1, 2, \ldots, n-1$. □

Claim 3. *Each I_i contains a vertex which is adjacent to each H_j* $(j \in \{1, 2, \ldots, n\}, j \neq i)$.

Fig. 1.

Suppose, without loss of generality, that I_n does not contain a vertex which is adjacent to each H_i $(i = 1, \ldots, n-1)$. Let I_{n_i} denote those vertices in I_n which are adjacent to no vertex in H_i. Clearly $I_n = I_{n_1} \cup I_{n_2} \cup \cdots \cup I_{n_{n-1}}$. Each $A_i \cup I_{n_i} = I_i \cup H_i \cup I_{n_i}$ $(i = 1, 2, \ldots, n-1)$ is an independence set. Thus $a(1) + \cdots + a(n-1) \geqslant |V(G)| - |H_n|$, consequently $a(n) \leqslant |H_n|$. Since I and H_n form a 2-colorable subgraph of G, we have $a(2) \geqslant |H_n|$. Thus $a(2) \geqslant a(n)$, a contradiction. □(Theorem 9)

The converse of this theorem is not true. For example, we may take G as in Fig. 1.

Theorem 10. *Let G and H be graphs. Then*

$$\operatorname{ncds}(G \times H) \geqslant^* \operatorname{ncds}(G) \quad \textit{and} \quad \operatorname{ncds}(G \times H) \geqslant^* \operatorname{ncds}(H).$$

Proof. We only need to prove that $\operatorname{ncds}(G \times H) \geqslant^* \operatorname{ncds}(G)$. Let $\operatorname{cds}(G) = (a(1), a(2), \ldots, a(t), \ldots, a(n))$, $\alpha(t) = a(1) + \cdots + a(t)$ $(t = 1, 2, \ldots, n)$; $\operatorname{cds}(G \times H) = (b(1), b(2), \ldots, b(t), \ldots, b(n))$, $\beta(t) = b(1) + \cdots + b(t)$. Let $I_t = \{x_1, \ldots, x_{\alpha(t)}\}$ $(1 \leqslant t \leqslant n)$ be the vertices of the maximum t-colorable subgraph of G. Then $\{x_1\} \times V(H) \cup \{x_2\} \times V(H) \cup \cdots \cup \{x_{\alpha(t)}\} \times V(H)$ are the vertices of a t-colorable subgraph of $G \times H$. Thus

$$\beta(t) \geqslant \sum_{i=1}^{\alpha(t)} |\{x_i\} \times V(H)| = \alpha(t)\,|V(H)| \quad (t = 1, \ldots, n-1),$$

and

$$\beta(n) = \sum_{i=1}^{\alpha(n)} |\{x_i\} \times V(H)| = |V(G)|\,|V(H)|.$$

Therefore

$$\frac{\beta(t)}{|V(G \times H)|} \geqslant \frac{\alpha(t)\,|V(H)|}{|V(G)|\,|V(H)|} = \frac{\alpha(t)}{|V(G)|} \quad (t = 1, 2, \ldots, n-1),$$

and

$$\frac{\beta(n)}{|V(G \times H)|} = 1 = \frac{\alpha(n)}{|V(G)|}.$$

i.e., $\operatorname{ncds}(G \times H) \geqslant^* \operatorname{ncds}(G)$. □

Corollary 11. *Let G be a graph, and G^k be recursively defined by $G^1 = G$, $G^k = G^{k-1} \times G$. Then $\lim_{k \to \infty} \operatorname{ncds}(G^k)$ exists.*

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